Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that an irreducible Markov chain on a countable state space with one recurrent state is recurrent (i.e. all its states are).

T2. A stochastic process X is called **strictly stationary** if for all $n \in \mathbb{N}$ and all $t_1 < t_2 < \cdots < t_n$ the distribution of $(X_{t_1+s}, \ldots, X_{t_n+s})$ does not depend on s.

Let X be a Markov chain on S with transition probability $p_t(\cdot, \cdot)$ and starting distribution π. Show that X is strictly stationary if and only if π is invariant.

T3. Let X be an irreducible and recurrent Markov chain. Show that every non-negative harmonic function for X is constant.

T4. Let X be a Markov chain on a finite set S with transition probability $p_t(\cdot, \cdot)$ and Q-matrix $q(\cdot, \cdot)$. Let π be a strictly positive measure on S. Show that π is reversible if and only if

$$\pi(x)q(x,y) = \pi(y)q(y,x) \quad \forall x, y \in S.$$

(N.B. this result also holds for countable S.)

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Homework exercises:

H1. Show that an irreducible Markov chain on a countable state space with one positive recurrent state is positive recurrent (i.e. all its states are).

H2. (a) Show that an irreducible recurrent Markov chain on a countable state space has an invariant (not necessarily normalizable) measure.

(b) Show that an irreducible Markov chain on a countable state space has an invariant distribution if and only if it is positive recurrent.

H3. Let $p \in (0,1)\backslash\{\frac{1}{2}\}$. We consider the asymmetric random walk on $\mathbb{Z}$ defined by the Q-matrix

$$q(x, x-1) = 1-p, \quad q(x,x) = -1, \quad q(x, x+1) = p, \qquad \forall x \in \mathbb{Z}.$$

(a) Find two invariant measures for the above process such that both measures are not multiples of each other.

Hint: For the Poisson distribution we have the following estimate:

$$\mathbb{P}(\mathrm{POI}(\lambda) \geq k) \leq \exp\left(-k \log\left(\frac{k}{\lambda e}\right) + \lambda\right).$$

(b) Is the asymmetric random walk on $\mathbb{Z}$ transient or recurrent?

Deadline: Monday, 9.12.17